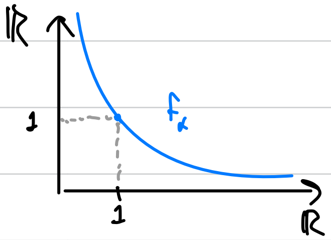
Math 565: Functional Analysis
Lecture 4

Discussion of L^p vs L^q when $p<q$. Let $1 \le p < q < \infty$. Fix $\alpha > 0$ and let $f_\alpha : (0,\infty) \to (0,\infty)$ be the function $x \mapsto \frac{1}{x^\alpha}$. Then Riemann integration shows that $\mathbb{1}_{(0,1)} \cdot f_\alpha \in L^p(\mathbb{R},\lambda) \iff p < \frac{1}{\alpha}$ and $\mathbb{1}_{[1,\infty)} \cdot f_\alpha \in L^p(\mathbb{R},\lambda) \iff p > \frac{1}{\alpha}$. So for $p<q$, the functions in L^q blow up faster locally, while functions in L^p decay slower at ∞. The first case only matters when there are arbitrarily small measure sets (e.g. atomless) while the second phenomenon happens in any infinite measure space.



Prop. Let $0 < p < q \le \infty$.

(a) $\ell^p(X) \subseteq \ell^q(X)$ for any set X. In fact, $\|f\|_q \le \|f\|_p$ for all $f : X \to \mathbb{C}$.

(b) $L^q(X,\mu) \subseteq L^p(X,\mu)$ for any finite measure space (X,μ). In fact, $\|f\|_p \le \mu(X)^{\frac{1}{p}-\frac{1}{q}} \|f\|_q$. In particular, when (X,μ) is a prob. space, then $\|f\|_p \le \|f\|_q$.

L^p norm via integration against L^q. Let $1 \le p \le \infty$ and let $q \ge 1$ be its conjugate exponent, i.e. $\frac{1}{p} + \frac{1}{q} = 1$ (equiv. $p - 1 = p/q$). Let (X,μ) be a measure space and $f : X \to \mathbb{C}$ be a μ-measurable function such that:

(i) $\{f \neq 0\}$ is σ-finite, i.e. $= \bigcup^{\uparrow}_{n \in \mathbb{N}} X_n$ with $\mu(X_n) < \infty$.

(ii) $f \cdot g \in L^1(\mu)$ for all simple $g \in L^q(\mu)$ ($\iff \{g \neq 0\}$ has finite measure).

Then

$$\|f\|_p = \sup\left\{ \left|\int fg \, d\mu\right| : g \text{ simple and } \|g\|_q = 1 \right\} = \sup\left\{ |fg\, d\mu| : \|g\|_{q} = 1 \right\}.$$

Proof. By Hölder's ineq., $\left|\int fg\, d\mu\right| \le \int |f| |g| \, d\mu \le \|f\|_p \cdot \|g\|_q = \|f\|_p$ for all $g \in L^q(\mu)$, so $\sup\left\{ \left|\int fg\, d\mu\right| : g \text{ simple and } \|g\|_q = 1\right\} \le \sup\left\{ |fg\, d\mu| : \|g\|_{q} = 1\right\} \le \|f\|_p$ and we need to show $\|f\|_p \le \sup\left\{ \left|\int fg\, d\mu\right| : g \text{ is simple with } \|g\|_q = 1\right\}$.

We first make the condition in the supremum less restrictive, leaving its proof for HW.

Claim. $\sup\{|\int fg\,d\mu| : g \text{ simple and } \|g\|_q = 1\} = \sup\{|\int fg\,d\mu| : \|g\|_q = 1,\ g \text{ bdd},\ \mu\{g \neq 0\} < \infty\}$.

We also leave for HW the proof for $p = \infty$ and assume $p < \infty$ here.
If we knew that f is bdd, has finite measure support and is in L^p, then as g we could take
$$g := \frac{|f|^{p-1} \cdot \overline{\operatorname{sgn} f}}{\|f\|_p^{p-1}}.$$
Then, recalling that $p-1 = p/q$, we would have:

(a) $\|g\|_q = \frac{1}{\|f_p\|^{p/q}} \cdot \left(\int |f|^{\frac{p}{q}\cdot q}\,d\mu\right)^{\frac{1}{q}} = \frac{1}{\|f_p\|^{p/q}} (\|f\|_p^p)^{\frac{1}{q}} = 1$;

(b) $\int fg\,d\mu = \frac{1}{\|f\|_p^{p-1}} \int |f|^p\,d\mu = \frac{1}{\|f\|_p^{p-1}} \|f\|_p^p = \|f\|_p$.

so $\|f\|_p = \int fg\,d\mu \leq \sup\{\dots\}$. However, we don't have that f is like that (bdd, finite measure support and in L^p), so we approximate f by such functions. Let $(\tilde{f}_n)$ be a sequence of simple functions such that $\tilde{f}_n \to f$ pointwise and $|\tilde{f}_n| \leq |f|$. Then $f_n := \mathbb{1}_{X_n} \tilde{f}_n$ are still simple, $f_n \to f$ pointwise and $|f_n| \leq |f|$, but also $\{f_n \neq 0\} \subseteq X_n$ so it has finite measure.
Take $g_n := \frac{|f_n|^{p-1}\ \overline{\operatorname{sgn} f}}{\|f_n\|_p^{p-1}}$. The g_n are not simple (because of $\operatorname{sgn} f$) but they are bdd and have finite measure support. Also, by the same calculations as above, we have

(a') $\|g_n\|_q = 1$;
(b') $\int |f_n||g_n|\,d\mu = \|f_n\|_p$;
(c') $fg_n = |f||g_n|$

Thus, applying MCT to $|f_n|^p \nearrow |f|^p$, we get
$$\|f\|_p = \lim_n \|f_n\|_p = \lim_n \int |f_n||g_n|\,d\mu \leq \lim_n \int |f||g_n|\,d\mu = \lim_n \int f g_n\,d\mu$$
$$\leq \sup\{|\int fg\,d\mu| : \|g\|_q = 1,\ g \text{ bdd},\ \mu\{g \neq 0\} < \infty\}. \qquad \square$$

Minkowski's inequality for integrals of functions. Let (X, μ) and (Y, ν) be σ-finite measure spaces and $1 \leq p \leq \infty$. Let $f : X \times Y \to \mathbb{C}$ be a $\mu \times \nu$-measurable function such that

$y \mapsto \|f(\cdot,y)\|_p$ is in $L^1(\nu)$ (in particular, $f(\cdot,y) \in L^p(\mu)$ for ν-a.e. $y \in Y$). Then $f(x,\cdot) \in L^1(\nu)$ for a.e. $x \in X$ and

$$\left\| \int f(\cdot,y)\, d\nu(y) \right\|_p \leq \int \|f(\cdot,y)\|_p \, d\nu(y). \qquad (*)$$

Remark. Minkowski's original inequality $\|f+g\|_p \leq \|f\|_p + \|g\|_p$ is a special case of this with $Y := 2$ and $\nu :=$ counting measure.

Proof. It suffices to prove the statement for $|f|$ since $f(x,\cdot) \in L^1(\nu) \iff |f(x,\cdot)| \in L^1(\nu)$ and $\int f(\cdot,y)\, d\nu(y) \leq \int |f(\cdot,y)|\, d\nu(y)$. Thus, we may assume WLOG that $f \geq 0$, so $\int f(x,y)\, d\nu(y)$ is defined and $(*)$ would automatically imply that $f(x,\cdot) \in L^1(\nu)$ for a.e. $x \in X$.

For $p = \infty$, $(*)$ is immediate because $\int f(x,y)\, d\nu(y) \leq \int \|f(\cdot,y)\|_\infty \, d\nu(y)$.

Suppose $p < \infty$. Denoting $F(x) := \int f(x,y)\, d\nu(y)$ $(*)$ becomes

$$\|F\|_p \leq \int \|f(\cdot,y)\|_p \, d\nu(y),$$

and we prove this by the calculation of $\|F\|_p$ via integration against $L^q(\mu)$. To apply this statement, we first verify that $Fg \in L^1(\mu)$ for all $g \in L^q(\mu)$.

Claim. $\|Fg\|_1 \leq \|g\|_q \cdot \int \|f(\cdot,y)\|_p \, d\nu(y) < \infty$ for all $g \in L^q(\mu)$.

Pf of Claim.

$$\int F \cdot |g| \, d\mu = \iint f(x,y)|g(x)| \, d\nu(y)\, d\mu(x) \overset{\text{Tonelli}}{=} \iint f(x,y)|g(x)| \, d\mu(x)\, d\nu(y)$$

$$[\text{Hölder}] \leq \int \|f(\cdot,y)\|_p \cdot \|g\|_q \, d\nu(y) = \|g\|_q \cdot \int \|f(\cdot,y)\|_p \, d\nu(y) < \infty. \quad \square \text{ (Claim)}$$

Thus, for $\|g\|_q = 1$, $\left| \int Fg \, d\mu \right| \leq \|Fg\|_1 \leq \int \|f(\cdot,y)\|_p \, d\nu(y)$, so

$$\|F\|_p = \sup \left\{ \left| \int Fg \, d\mu \right| : \|g\|_q = 1 \right\} \leq \int \|f(\cdot,y)\|_p \, d\nu(y). \qquad \square$$